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THE GEOMETRICAL ANALYSIS OF GRASSMANN AND ITS CONNECTION WITH LEIBNIZ'S CHARACTERISTIC.

§ I.

BY a curious turn of fate Grassmann wrote, in the introduction to his "Geometrical Analysis," concerning Leibniz's early work on the same subject, words which were to apply with prophetic force to his own *Ausdehnungslehre*. "When the special power of a genius. . . .is so revealed that he is able to grasp and extend the ideas toward which the development of his time is directed, and so appears representative of his period, then that power shows itself still more remarkable when it can seize ideas in those realms of thought in advance of their day and forecast for hundreds of years the line of their development. While ideas of the first kind are often developed simultaneously by the outstanding spirits of the age, as when both Leibniz and Newton founded the differential calculus—a certain stage of fruition being reached—ideas of the latter kind appeared as the special characteristic of the individual, the innermost revelations of his mind into which only a few elect contemporaries could enter and have a foreshadowing of the developments which were to spread from them in the future. While the first received great applause and aroused movement in their own day, because they represent the summit of the epoch, the others for the most part fall without effect in the contemporary period since they are only understood by a few, and then only partially. Often afterward does such thought become the seed of a rich harvest. That this great idea of Leibniz—namely, the idea of a true geometrical analysis—belongs to this preparatory and, as it were, prophetic class cannot be doubted for a moment. It has also shared the fate of such. Indeed by a special ill favor of circumstances it has remained hidden far beyond the time when it might have had a powerful influence. For even before it was brought out of its hiding place by Uylenbroek, paths toward a similar analysis had been made in other ways."

At the time when these words were written Grassmann could have had no idea of the disappointment which was to come to him in the neglect of his own work. The first edition of the *Ausdehnungslehre,* or theory of extended magnitudes, had been published in 1844 and had received no attention from mathematicians with the exception of a few individuals. Grassmann, however, believed that recognition was only a matter of time and sought to bring out the importance and applicability of the new analysis. For the year 1845 (but extended to 1846 to coincide with the two hundredth anniversary of Leibniz's birth) the Jablonowski Society of Leipsic set a prize essay demanding the restatement or further development of the geometrical calculus discovered by Leibniz or the setting up of a similar calculus; and the award was made to Grassmann for the essay, printed in 1847, from which I made the above quotation. This was the first and the only acknowledgment of the value of his work which he received from mathematicians until long after many of the ideas he formulated had been reached and applied by other methods and other thinkers.

I have laid stress on the similarity of treatment meted out to the fundamentally important work of the two men because I believe that in some elements of its explanation lies the clue to unravel the difficulties of their subject matter and connection with each other. The more general aims of both Leibniz and Grassmann were the same—the setting up of a convenient calculus or art of manipulating signs by fixed rules, and of deducing therefrom true propositions for the things represented by the signs, for use as a generalized mathematics. In each case their geometrical calculus was a particular application to geometry of a wider calculus for which each desired more than mere applicability to mathematics.

In a letter to Arnauld, dated January 14, 1688, Leibniz writes[1]: "Some day, if I find leisure, I hope to write out my meditations upon the general characteristic or method of universal calculus, which should be of service in the other sciences as well as in mathematics. I have definitions, axioms, and very remarkable theorems and problems in regard to coincidence, determination, similitude, relation in general, power or cause, and substance, and everywhere I advance with symbols in a precise and strict manner as in algebra. I have made some applications of it in jurisprudence." Similarly Grassmann[2] says: "By a general science of symbols (*Formenlehre*) we understand that body of truths which apply alike to every branch of mathematics, and which presuppose only the universal concepts of similarity and difference, connection and disjunction." The symbols are made so general as to be applicable to both logic[3] and mathematics, although in the *Ausdeh-*

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[1] George R. Montgomery (trans.), *Leibniz: Discourse on Metaphysics, Correspondence with Arnauld, and Monadology*, p. 241.

[2] *Ausdehnungslehre* of 1844, p. 2.

[3] The application of such a general science of symbols to formal logic was made by both H. Grassmann and his brother Robert.

nungslehre they are only applied to the domain of mathematics.[4]

It is clear that both Leibniz and Grassmann, but especially the former, claimed great scope for their calculus, a fact which tended to make their writings generalized and difficult to understand. In the preface to his *Universal Algebra* (1898) Professor Whitehead expresses his belief that lack of unity in presentation (which of course would be the tendency in dealing with a method applicable to many fields) discourages attention to such a subject. But that is not all. A new mathematical method, to make itself known, has to appeal in the main to mathematicians and not to philosophers. So that a wide and philosophical treatment is apt to be discounted by the ordinary man who thinks logic can be made to prove anything.

§ 2.

Before we condemn this attitude we must first of all inquire as to what exactly the common man means by the dangers of logic. What he really fears is not logic but fallacy. Without realizing it he distrusts a mechanical dexterity in reasoning because the attainment of truth depends not only on a facility in manipulating logical processes but also on the sifting of first principles. When Leibniz claims for his *characteristica universalis* or "universal mathematics," the germ of which he produced in his *De arte combinatoria* published when he was twenty, that "....there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other (with a friend as witness, if they liked): Let us calculate"—he is running counter to the plain man's knowledge that there are two parts of a logical process, the first the choosing of an assumption, the second the arguing upon it.

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[4] In the *Ausdehnungslehre,* however, are expressions directly applicable to logic, e. g., there is the generalized expression for the result of division $C + 0/B$ where $0/B$ is an indefinite form (p. 213)—an anticipation of Boole's use of $0/0$ to symbolize perfect indefiniteness, as pointed out by Venn in his *Symbolic Logic,* note p. 268 (2d ed.).

Now Leibniz realized of course that premises are required first, but he thought they could be obtained very simply. By analyzing any notion until it was simple he thought that all axioms or assumptions followed as identical propositions. Thus he was led, by his view of ideas, to believe that even the axioms of Euclid could be proved. So in his *New Essays,* "I would have people seek even the demonstration of the axioms of Euclid.... And when I am asked the means of knowing and examining innate principles, I reply....we must try to reduce them to first principles, i. e., to axioms which are identical, or immediate by means of definitions which are nothing but a distinct exposition of ideas." This is connected with his view that all our ideas are composed of a very small number of simple ideas, which together form an alphabet of human thoughts. But, as Couturat remarks,[5] there are many more simple ideas than Leibniz believed; and furthermore there is no great *philosophical* interest in such. "An idea which can be defined or a proposition which can be proved, is only of subordinate philosophical interest."[6] It is precisely the business of philosophy to deal with the primitive, intuitive assumptions on which any calculus must be based.

So the plain man is to some extent justified in his mistrust of the uncritical application of a calculus.

§ 3.

It is very necessary, however, to see exactly what is, and what is not, here granted to the plain man. It is true that in using a calculus we must be careful not to over-emphasize the results at the risk of forgetting the premises from which they have been obtained. But that being admitted, thus making the final development of the universal characteristic a matter not of philosophy but of a sort of generalized mathematics of which formal logic[7] and geometry are special cases, it does not follow that there must be limits to the applicability of the calculus in these spheres. Yet that is what the modern representative of our plain man asserts. His criticism of a logical calculus has put on a more philosophical form, but remains essentially the same. Henri Poincaré may justly, I think, be taken as such a representative. For he says, "I appeal only to unprejudiced people of common sense.... they [the logisticians] have shown that mathematics is entirely reducible to logic, and that intuition plays no part in it whatever."[8] This belief led Poincaré to the view that, since he knew from his own experience as a mathematician of great insight the important part intuition plays in mathematical discovery, therefore the *nature* of mathematics cannot be logical.

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[5] L. Couturat, *La logique de Leibniz,* p. 431.

[6] B. Russell, *The Philosophy of Leibniz,* p. 431.

This reasoning is founded on a very common fallacy which I will call the genetic error—the error, namely, which lies in the assumption that the origin of a thing in some way determines its nature.[9] If this assumption is made it follows that since intuition plays a part in discovery, the nature of mathematics cannot be purely formal, and therefore it cannot be expressed in terms of symbolic logic. Now all such references to the origins of mathematics are irrelevant. Once the premises have been made, and that is where intuition comes in, symbolic logic is merely "an instrument for economizing the exertion of intelligence."[10] The mind, being relieved of unnecessary work by a good symbolism, is set free to attack more difficult problems; for as Professor Whitehead says,[11] "Operations of thought are like cavalry charges in a battle—they are strictly limited in number." Nor is that the only advantage of this modern development of Leibniz's universal mathematics. It "has the effect of enlarging our abstract imagination and providing an infinite number of possible hypotheses to be applied in the analysis of any complex fact."[12] And so it lends itself to the production of just such novel fundamental hypotheses as are needed in subjects like the dynamics of relativity.

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[7] Leibniz himself foresaw this development carried out by Boole, Peano, Frege, Whitehead and Russell and their school of symbolic logicians. In fact he made discoveries in this field but did not publish them because they contradicted certain points in the traditional doctrine of the syllogism. In some points he even advanced beyond Boole (See Couturat, *op. cit.*, p. 386).

[8] *Science et Méthode*, p. 155; cf. also C. J. Keyser in *Bull. Amer. Math. Soc.*, Jan. 1907, pp. 197, 198.

[9] This error has been very common in philosophy. It underlies much argument against rationalism, denying that knowledge reached empirically can be anything other than empirical. (Cf. Leibniz, *New Essays*, IV, 1 § 9, against Locke.) It is at the basis of many criticisms leveled against any generalization of number, since the idea of number arose from perceptual experience. It vitiates pragmatism, which inquires into the causes of our judging things to be true in order to get at the nature of truth. (See B. Russell, *Philosophical Essays*, p. 110.)

So finally, we must say of the symbols of a universal calculus what Hobbes said of words, "They are wise men's counters, they do but reckon by them; but they are the money of fools." Yet it must be recognized that when it is confined to dealing with mathematics in its widest sense (taken to include formal logic),—within the limits imposed on his own calculus by Grassmann, in fact,—it serves as a powerful and legitimate tool.

§ 4.

This discussion of the neglect and mistrust of mathematicians for the generalized calculus of both Leibniz and Grassmann has, I hope, shown what the nature of such a calculus is. Moreover, it accounts for the long period which elapsed before their fruitful application of these methods of calculation to special fields obtained the notice they deserved.

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[10] W. E. Johnson in *Mind,* N. S., Vol. I, pp. 3, 5. Cf. Stout, "Thought and Language," *Mind,* April, 1891.

[11] *An Introduction to Mathematics,* Home Univ. Library, p. 59. See also P. E. B. Jourdain in *The Monist,* Jan. 1914, p. 141.

[12] B. Russell, *Our Knowledge of the External World,* pp. 58, 242.

The particular application we are here concerned with is that to geometry. In a letter to Huygens of September 8, 1679, Leibniz complained that he was not satisfied with the algebraic methods, and adds: "I believe that we must have still another properly linear geometrical analysis, which directly expresses *situm* as algebra expresses *magnitudinem.* And I believe I have the means for it, and that one could represent figures and even machines and movements in symbols, as algebra represents number or magnitude; I am sending you an essay which seems to me notable." This essay contained an account of his geometrical calculus in which the relative position of points is denoted by simple symbols and fixed without the help of the magnitude of lines and angles. It differs therefore from ordinary algebraic analytical geometry. The further development of this calculus was the subject of Grassmann's *Geometrische Analyse*[13] which we have already noted as being crowned by the Jablonowski Society. This was done on the recommendation of Möbius, who found in Grassmann's essay a generalization and extension of his own barycentric calculus.

We will now consider the geometrical calculus of Leibniz with a view to discovering if Grassmann's development of it has fulfilled in any way Leibniz's hopes of its ultimate importance.

§ 5.

The letters and papers of Leibniz in which he deals with his project of a geometrical calculus are many, and

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[18] This treatise develops some of the subjects which Grassmann had intended for a second part of the 1844 *Ausdehnungslehre,* which was never written.

spread over a considerable period of time.[14] The most important is the *Characteristica geometrica,* a sketch of the notion which he made for fear it should be lost if he found himself unable to develop it. The essay enclosed in the letter to Huygens in 1679 was an extract from this. From these writings it seems clear that the starting point was his conviction of the imperfection of algebra as the logical instrument of geometry. Thus, "Algebra itself is not the true characteristic of geometry, but quite another must be found, which I am certain would be more useful than algebra for the use of geometry in the mechanical sciences. And I wonder that this has hitherto been remarked by no one. For almost all men hold algebra to be the true mathematical art of discovery, and as long as they labor under this prejudice, they will never find the true characters of other sciences." It must be noted that algebra is here used by Leibniz in its ordinary sense, not as a general term for any calculus.

He saw that analytical geometry only expressed geometrical facts in a complicated and roundabout manner. A figure such as the circle is not defined by its internal relations, but by reference to its relations to arbitrary coordinates. So a set of magnitudes foreign to the figure are introduced and obscure the purely geometrical relationships. Further, to reduce relations of position to relations of size presupposes Thales's theorem about similar triangles and the theorem of Pythagoras.[15] In other words analytical geometry is thus made dependent on synthetic. The analysis not being pushed far enough, it has not the logical perfection which belongs to a purely rational science.[16] He realized the want of rigor and generality of intuitive methods, but dreamed of a method which would be completely analytical and rational while still possessing all the advantages of a synthetic method.

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[14] An interesting bibliography of them together with an account of the main ideas which inspired and directed his search for a geometrical characteristic is given in Couturat, *La logique de Leibniz,* 1901.

[15] *Characteristica geometrica,* § 5.

[16] Cf. his letter to Bodenhausen.

In this his aim was similar to that which he had in mind for his universal characteristic, which was to be a logical calculus replacing concepts by combinations of signs, and which furthermore was not merely to furnish demonstrations of propositions but to be the means of discovering new ones. So, in like manner, his geometrical calculus was to combine analysis with guidance of the intuition.[17] A fusion of analysis and synthesis being made, the divorce between calculation and construction would disappear. "This new characteristic.... will not fail to give at the same time the solution, construction, and geometrical demonstration, the whole in a natural manner and by an analysis."[18] It is clear that the final goal was a science of form of very wide application.[19] This aim we must distinguish carefully from the manner in which he attempted to realize it.

As Grassmann points out in the introduction to his "Geometrical Analysis," this distinction between the distant goal and his attempt toward a new characteristic which he connects with it to render the thought more realizable, is recognized fully by Leibniz. Although the characteristic he provided will be seen to be only a small first step toward the goal he had set himself, yet he had estimated the essential advantages of a final geometrical analysis to an extraordinary completeness. Grassmann says: "Just this eminent talent of Leibniz of being able to foresee in presentiment a whole series of developments without being able to work it out and without dismembering and dissecting it, yet to make it present to himself with prophetic mind and to recognize the importance of its consequences—it is just this talent which led him to such great discoveries in almost all domains of knowledge."

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[17] Leibniz conjectured that the ancients had some natural and spontaneous analysis of this kind resting on the abstract relations of figures, which underlay and helped their synthetic methods. (*De analysi situs.*)

[18] Letter to Huygens.

[19] *Ibid.* "I believe that one could handle mechanics by these means almost like geometry."

§ 6.

Leibniz founds his fundamental definitions on congruence, which means the possibility of coincidence. He represents points whose positions are known by the first letters of the alphabet, and those which are unknown or variable by the last letters. Any two combinations of corresponding points are said to be congruent if both can be brought to coincide without the mutual position of the points being changed in either of the two combinations; so that every point of one combination covers a corresponding point in the other. Congruence (geometrical equality) is a union of two relations—similarity and equality (quantitative equivalence).

All *points* are equal and similar, so all points are congruent.[20] Hence if we use $\equiv$ for congruence, the expression $a \equiv x$, where a is fixed and x is variable, is a definition of space.

It must be noticed that in defining figures by congruence the *axiom of congruence* or *free mobility*[21] must be postulated. If we do this, $ax \equiv bc$ represents a sphere of center a and radius bc.

Also, $ax \equiv bx$ represents a plane which bisects ab perpendicularly.

The above can be taken as the definition of the sphere and the plane respectively. Again $ax \equiv bx \equiv cx$ gives the locus of the center of all spheres which pass through a, b, c; and so it is a straight line.

If $\qquad ax \equiv ac \qquad$ and $\quad bx \equiv bc$

they together give the common trace of two spheres.

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[20] *Characteristica geometrica.*

[21] See B. Russell, *Foundations of Geometry,* Cambridge, 1897.

Combined they are written $abx \equiv abc$. This therefore represents the locus of points whose distances from the points a, b are the same as the distances of c from a, b. That is, it is a circle.

The economical nature of the symbolism is shown by the fact that if we take this as a definition of the circle, it does not imply the idea of the straight line or the plane; nor does it require (as the circle defined by an algebraic equation) that the center of the circle must be known.

As an example of a proof consider the proposition that *the intersection of two planes is a straight line.*

Let $ay \equiv by$ be one plane
and $ay \equiv cy$ be the other.

Then $ay \equiv by \equiv cy$, and this we saw above to be the form of the congruences representing a straight line.

In these examples is a faint foreshadowing of the side by side development of construction, proof and analysis. And since all kinds of spatial relationships can be developed from the line and the sphere, the method is capable of wide extension.

§ 7.

There are several obvious defects in it, however. These appear at once if we attempt by means of it to solve the fundamental problem in geometry of finding the expression for a straight line passing through two given points. Leibniz had previously attacked the problem only to find himself involved in difficulties.[22]

Grassmann's treatment is as follows: We saw above the expression for a straight line was

$$ax \equiv bx \equiv cx.$$

If we now take three auxiliary points, a', b', c', which are not in a straight line, and write

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[22] Couturat gives a clear account of this, *op. cit.*, pp. 420-427.

$$\begin{cases} a'x \equiv b'x \equiv c'x \\ a'a \equiv b'a \equiv c'a \\ a'b \equiv b'b \equiv c'b, \end{cases}$$

then together these congruences represent the required straight line through a, b, as the locus of x.
Combining the last two we get

$$\begin{cases} a'x \equiv b'x \equiv c'x \\ aba' \equiv abb' \equiv abc'. \end{cases}$$

This then expresses that the auxiliary points lie on the circle the plane of which is cut at its center by the line *ab* at right angles.

If this expression is to have the necessary simplicity, it must be possible to eliminate the arbitrary auxiliary points which have nothing to do with the nature of the problem, and to combine the group of formulas into one. That being impossible, the characteristic has failed to serve its purpose.

Indeed the failure of the method followed at once from the choice of congruence as the fundamental relation. For, as we have seen, this complex relation contains a quantitative element, and so prevents any freeing of geometry from considerations of magnitude. In fact, as the above expression for the line through *ab* shows, we are still left with arbitrary coordinates. Further, in this system there is also ambiguity, as Couturat has shown.[23] In other words the analysis had not gone far enough. If what remained of magnitude had been eliminated—not merely by taking the relation of similarity, for Leibniz had himself shown that to imply metrical relations[24]—but by reducing figures to their projective properties and relations, at least a real geometry of position[25] would have followed. But such a projective geometry, while satisfying Leibniz's desire to eliminate algebraic methods from geometry, would not have been a geometrical calculus with points as elements. Nor could it have had the wide application which he sought for in his calculus; for if it was to be applicable to mechanics and physics, it must at some point have been susceptible of metrical development.

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[23] *Ibid.*, p. 428, note 2.

[24] *"Elementa Nova Matheseos Universalis."*

[25] Developed by Staudt, *Geometrie der Lage*, 1847.

Now, throughout our discussion we have seen that Leibniz was seeking for a characteristic particularly applicable to geometry but akin to his universal characteristic. At the end of the letter to Huygens he says: "I believe it is possible to extend the characteristic to things which are not subject to imagination." In other words he was seeking a formal calculus, an abstract mathematics lying at the base of geometry and applicable not only to it but also to logic. Now Grassmann had already developed such a science of form in his *Ausdehnungslehre* of 1844. So when the Jablonowski Society announced the subject of their prize essay he took the opportunity of expounding his science of extensive magnitudes, not as he had originally derived it, but starting from Leibniz's characteristic.

§ 8.

When he had proved the insufficiency of the relation of congruence as Leibniz had left it, he tried to give it a form in which substitution would be possible. What are congruent to the same thing are congruent to each other, but that does not mean that we can in a general way place instead of a given term in a congruent expression one congruent to it. So substitution is not possible. This can be seen at once. All points are congruent. Therefore, if one could substitute the congruent, one could place *abc* congruent to every combination of three points—which is absurd.

Grassmann rightly regarded the fact that substitution was not possible as a serious defect in the calculus. So he inquired what equations would hold between the points a, b, c, d, e, f, if $abc \equiv def$.

There must be some function f such that, when the above holds,

$$f(a, b, c) \equiv f(d, e, f).$$

So he was led to the general linear relation of collinearity.

Now in the *Ausdehnungslehre* Grassmann had reached the fruitful idea of a true geometrical multiplication which has the peculiarity that if any two factors of the product are interchanged the sign of the product is changed, that is,

$$\mathrm{AB} = -\mathrm{BA}.$$

This *combinatory* multiplication enabled him now to give an intrinsic definition of geometrical figures in terms of points, and so to accomplish what Leibniz had failed to do. Thus the product ab determines the straight line between the points a, b; the product of three points determines the plane, and so on. But since the product is non-commutative these figures when so defined have a sense represented by the signs + or —. Furthermore, he conceived the notion of using these products to express not only relations of position but also of magnitude. So that the same analysis which gave a geometry of position also gave, side by side and without confusion, a metrical geometry. In making this step he had to define (§ 3, *Geometrische Analyse*) what he meant by a point magnitude. Each element (point, line, plane) has two aspects—its position in space, and its intensity. In the case of the point, this latter was represented by a positive or negative "mass."

By now defining a *line magnitude* as the combination ab of the point magnitudes a, b— the direction of which is through a and b, and the intensity of which can be defined; and also defining the *point magnitude* as the combination AB of two line magnitudes, the position of which is the intersecting point of A and B and the mass value of which can be made the subject of a definition—then by an assumption which makes $ab = O$ and $AB = O$ represent coincident lines and points, it is possible to write in the form of an equation every linear dependence.

Thus $(ab)(cd)e = O$ denotes that e is the intersecting point of ab and cd.

So the principle of collineation can be expressed, though cumbrously without further adaptation, by such combination equations.

In this way *equality* is made to include the two relations of identity of position and equality of intensity. So projective and metrical relations can be expressed in one form, and considered either separately or together.

§ 9.

It is impossible to follow Grassmann's development[26] further without setting up a technical symbolism, but it may easily be shown how brilliantly Leibniz's hopes of an analysis specially applicable to mechanics have been fulfilled.

In terms of this calculus the sum of n points is their mean point. If intensities are considered, the metrical relation follows. Thus if the intensities represent masses at the points the sum gives the center of gravity of the system—a point whose intensity will be the sum of the other intensities. If the intensities represent parallel forces acting at the point the sum gives the point of application of the resultant. The *barycentric calculus* of Möbius is thus included in this more general analysis.

Furthermore, the line magnitude of Grassmann expresses a force with exactitude. Composition of forces thus becomes the addition of line magnitudes. The general equations of dynamics can also be represented (§ 11, *Geo-*

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[26] Needless to say the above is a mere sketch of the beginning of Grassmann's "Analysis." In particular no mention is made of his distinction between *inner* and *outer* products.

metrische Analyse) by means of this calculus, as soon as certain modes of treating infinitesimals have been evolved.

Moreover the possibility of attaching a metrical coefficient to each point in space opens at once many fields of application in physics.

We must notice in addition that the "Geometrical Analysis" does not treat of the *quotients* of non-parallel stretches, a subject which leads to a calculus for dealing with powers, roots, logarithms and angles.

Grassmann can claim justly therefore, as he does, in the concluding remarks to this work, that his mode of treatment, if transferred to physics in general, would simplify the mathematical treatment in a splendid manner. He himself has shown the great advantages of the calculus in many fields. In the essay we have several times referred to, Leibniz wrote, "If it [the characteristic] were set up in the manner I conceive, one could construct in symbols, which would only be the letters of the alphabet, the description of any machine.... One could by these means make exact descriptions of natural objects."

As an example of such descriptive power Grassmann mentions his application of the calculus to crystallography (cf. *Ausdehnungslehre* of 1844, § 171).

§ 10.

Apart from the adaptability of the geometrical calculus to different provinces, there are other good reasons for believing that it realizes the ideal toward which Leibniz looked forward.[27] Grassmann's claim put forward in his concluding remarks will, I think, be granted by any one willing to master the symbolism sufficiently to understand any of his theorems. "As in the analysis demonstrated here every equation is only the expression, clothed in the form of the analysis, of a geometrical relation, and this relation expresses itself clearly in the equation without being obscured by arbitrary magnitudes—as for example the coordinates of the usual analysis—and therefore can be read off from it without further trouble; and as further every form of such equation is only the expression of a corresponding construction, then it follows that as a matter of fact, by means of the analysis here given, the solution of a geometrical problem results at the same time as the construction and the proof. As further nothing arbitrary ....need be introduced, the kind of solution must always be according to the nature of the problem; and as it is in the form of analysis, therefore a necessary one in which there can be no question of any seeking round for methods of solution." In other words the fusion of synthetic and analytic methods which Leibniz hoped for is fully accomplished.

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[27] Letter to Huygens: "Algebra is nothing but the characteristic of indeterminate numbers, or of magnitude. But it does not express exactly situation, angles and movement.... But this new characteristic....cannot fail to give at the same time the solution, the construction and the geometrical proof, the whole in a natural manner and by analysis. That is by determined ways."

It must be noted that in one respect Grassmann has not only realized the prophetic vision of Leibniz but also cleared away the inconsistency which vitiates his attempt at making his dream come true. For Leibniz, seeing that the *fundamental* analysis of geometry must rest on non-metrical relations, yet desired its *final* application to mechanics and natural science, in which metrical relations are all important. So he was led to a half-hearted attempt at non-metrical analysis by means of a relation—congruence—which, while showing the way to a geometry not based on algebra, yet failed itself to travel far in that direction. The special merit of Grassmann has been to found a geometrical analysis free of magnitude and yet so to develop it that metrical considerations may be introduced without disturbing the form of that analysis. Projective geometry, therefore, only partly fulfils Leibniz's hopes;

their complete realization is found in Grassmann's theory of extension.

§ 11.

We began our discussion of the relation between Grassmann's calculus and the characteristic of Leibniz by an analysis of the manner in which their work has been received by the average mathematician. It seems to me that we can profitably return to these historical considerations for a moment, and look at them from another view-point.

There is some reason, as I have tried to show elsewhere,[28] for citing lack of historical perspective on the part of mathematicians as the cause of the unsympathetic attitude commonly taken up in regard to work of philosophical breadth; and that if more regard were paid to historical development in mathematical education wider and more penetrating vision would result. The position taken up is well expressed by Branford[29]: "The path of most effective development of knowledge and power in the individual coincides, in broad outline, with the path historically traversed by the race in developing that particular kind of knowledge and power." At the same time, however, we must realize that, if we alter our attitude to this slightly, and regard it not from the point of view of the educationalist but from that of the original worker himself, obsession with origins seems inevitably to lead to what I have called above the genetic error. The effective point of departure in attaining knowledge of geometry may be from such empirical and utilitarian experiments as form its historical origin. But that must not be allowed to create an atmosphere hostile to any recognition of the *a priori* and formal nature of that science.

Furthermore the historical method may lead to a certain *ex cathedra* manner, a reliance on authority and tradition. It is this factor which especially concerns us in our attempt to see the work of Leibniz and Grassmann in true relation to each other and to mathematical thought. For Couturat points out[30] that what probably hindered Leibniz's development of his geometrical calculus and rendered abortive his attempt at its realization was the authority of Euclid. He says, "Why, amidst all the relations which Leibniz catalogued, did he give preference to the relation of congruence and neglect the relations of similarity, inclusion, situation, which serve to-day as the bases of quite new sciences[31] which he foresaw and would have been able to found? It is evidently because tradition, represented and embodied by the *Elements* of Euclid, limited geometry to the study of the metrical properties of space. Now the tradition is not explicable by any reason of *theoretical* order (considering that metrical relations are more complex and less general than projective relations) but solely by reason of historical and practical order."

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[28] "The Neglect of the Work of H. Grassmann."

[29] B. Branford, *A Study of Mathematical Education,* 1908.

I have already in the previous section shown that another explanation may be held of this clinging to a metrical relation by Leibniz. However that may be, the authority of the Euclidean tradition may have had some influence on his work in geometry, as the Aristotelian tradition had in his foreshadowing of a logical characteristic.[32] In fact we shall not be laying over-emphasis on the tendencies of an exaggerated reliance on historical method if we say that its final result is the attitude of the young critic in Shaw's play[33] who says, in effect, "Give me the name of the author and I'll tell you if it's a good play." If that critic held a university chair of historical criticism he would doubtless be able to find valid arguments for his position—for how

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[30] *La logique de Leibniz*, pp. 438-440. Russell however attributes Leibniz's failure to his holding the relational theory of space, *Mind*, 1903, p. 190.

[31] Theory of aggregates, modern *Analysis situs*, projective geometry, etc.

[32] See note 7 above.

[33] "Fanny's First Play."

(he might ask) can one judge competently without a complete set of data, and is not authorship an important datum? It is irritation at this standpoint which causes Mr. Bertrand Russell, whom I have heard speak very forcibly on the subject, inveigh against this hyper-historical method. But the objection can be stated in a much stronger form. "Erudition often does violence to inventive power: and the proof is that the modern discoverers of symbolic logic, Boole and his successors, have all ignored (and rightly) the example and precedent of Leibniz; it has even been remarked[34] that they have almost all been ignorant of one another, and if this ignorance has been a source of error, it has been above all a condition of originality."[35] Now it does not appear to me that the essential defect of such an extreme anti-historical attitude has been that it caused error. Staudt realized the ambitions of Leibniz in some degree in founding his projective geometry, and Grassmann in still further degree in creating his theory of extension, without knowing that their historical origins lay in his work. No great harm comes from this, although an original genius will, as a general rule, be less likely to be deflected from his way by the work of others than to find in them sources of stimulation. But to the mass of us, who form the bulk of mankind, narrowness is a mental blinker which hides the full splendor of the creations of genius. The real toll taken by historical ignorance is in neglect of originality, and the loss of power and influence consequent on it.[36]

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[34] J. Venn, *Symbolic Logic,* Introd., pp.29, 30.

[35] Couturat, *op. cit.,* p. 440.

[36] I have to thank Miss Vinvela Cummin for valuable help in translating the *Geometrische Analyse.*